Inference

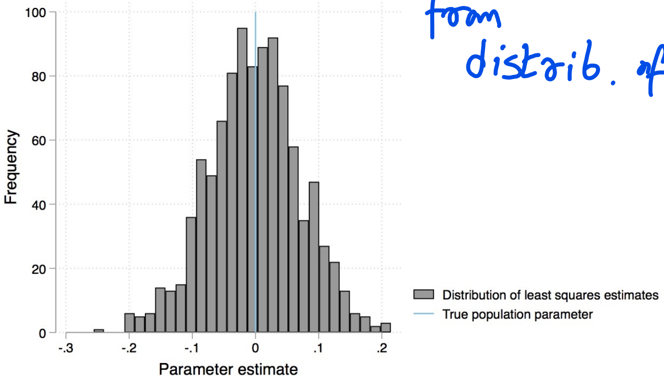
1. Sampling Distributions
2. Single Hypothesis - Single Parameter
   - Critical value
   - p-value
   - Confidence interval
3. Single Hypothesis - Multiple Parameters
4. Multiple Hypotheses

Sampling Distributions

- Can write

$\hat{\beta}_j = \beta_j +$ a term linear in u

- Distribution of $\hat{\beta}_j$ follows from distrib. of u



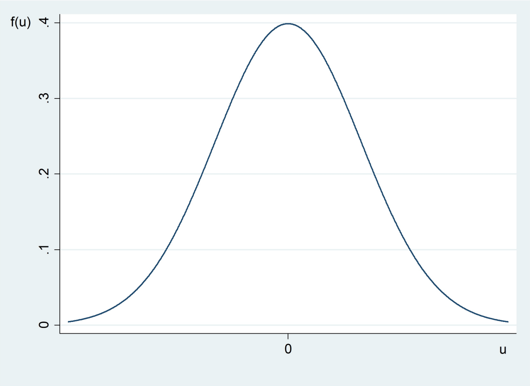
Causal Inference: The Mixtape

Sampling Distributions (cont.)

(MLR.6) Normality

u indep. of $(x_1, \ldots, x_k)$
with $E(u) = 0$
$Var(u) = \sigma^2$

$u \sim N(0, \sigma^2)$



Sampling Distributions (cont.)

- Under MLR.1 to MLR.6

$$\hat{\beta}_j \sim N(\beta_j, Var\left(\hat{\beta}_j\right))$$

- So

$$\frac{\hat{\beta}_j - \beta_j}{sd(\hat{\beta}_j)} \sim N(0,1)$$

Sampling Distributions (cont.)

we have $se(\hat{\beta}_j)$ instead of $sd(\hat{\beta}_j)$

- Using $\hat{\sigma}$ in place of σ

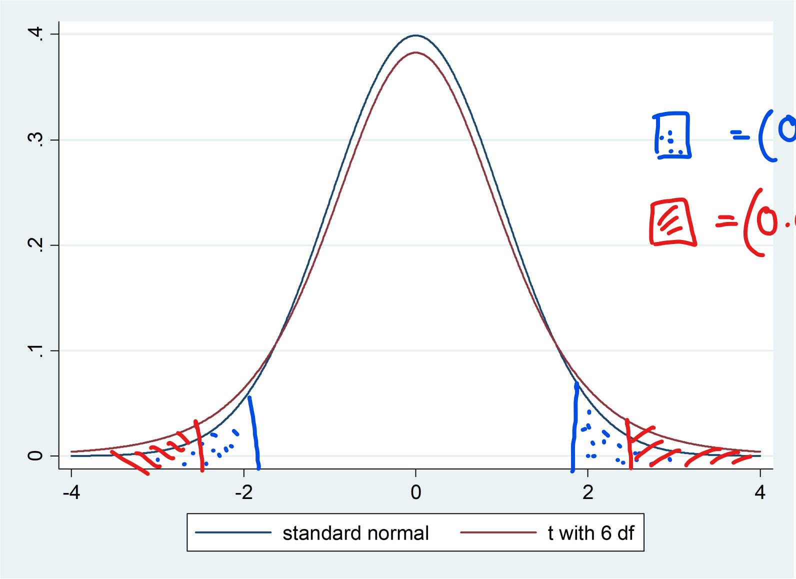
follows a t dist. with $(n-k-1)$ degrees of freedom

$$\frac{\hat{\beta}_j - \beta_j}{se(\hat{\beta}_j)} \sim t_{n-k-1} = t_{df}$$

- As $df \to \infty$

$$t_{n-k-1} \to N(0,1)$$

Sampling Distributions (cont.)



= (0.025)2

= (0.025)2

Single Hypothesis - Single Parameter

- Test hypotheses about β_j
- Null (H_0) and alternative hypotheses (H_1)

https://xkcd.com/892/

Single Hypothesis - Single Parameter (cont.)

Two-tailed test

- Model

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_k x_k + u$$

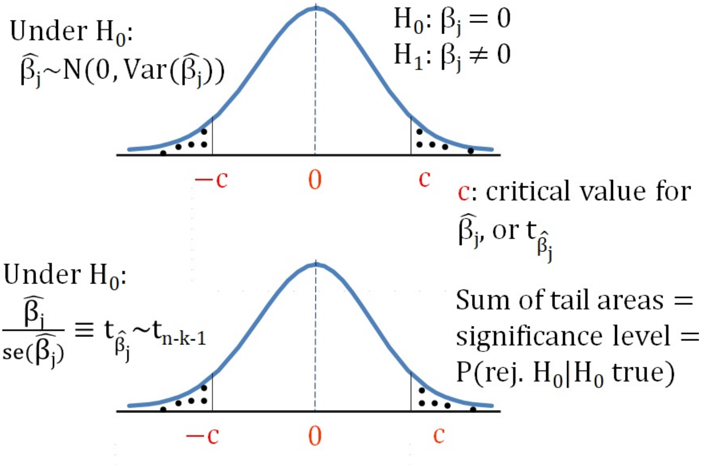
- Test whether x_j has a non-zero effect after controlling for all other indep. variables

$$H_0 : \beta_j = 0$$
$$H_1 : \beta_j \neq 0$$

- t statistic, or t ratio

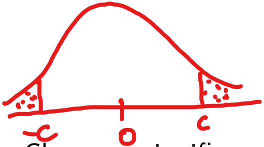
$$t_{\hat{\beta}_j} \equiv \frac{\hat{\beta}_j}{se(\hat{\beta}_j)}$$

Single Hypothesis - Single Parameter (cont.)



Single Hypothesis - Single Parameter (cont.)

$\boxed{\therefore} = 0.05$ (e.g.)



- Choose a significance level : $P(\text{rej. } H_0 \mid H_0 \text{ true})$
- Reject H_0 if $|\hat{\beta}_j|$ or $|t_{\hat{\beta}_j}|$ is sufficiently large else fail to rej. H_0
- Rejection rule

$$\left| t_{\hat{\beta}_j} \right| > c$$

(never accept!)

- Critical values
  - Table G.2: t distribution
  - Table G.1: standard normal distribution

Single Hypothesis - Single Parameter (cont.)

$H_0: \beta_1 = 0$

$H_1: \beta_1 \neq 0$

$wage = \beta_0 + \beta_1 educ + u$

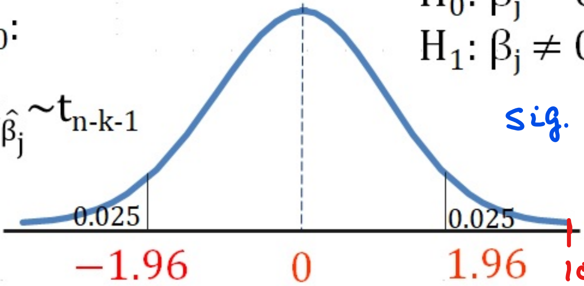
$\hat{\beta}_1 = 0.541 \qquad se(\hat{\beta}) = 0.053 \qquad t_{\hat{\beta}_1} = 10.167$

Under H_0:

$$\frac{\hat{\beta}_j}{se(\hat{\beta}_j)} \equiv t_{\hat{\beta}_j} \sim t_{n-k-1}$$

$H_0: \beta_j = 0$

$H_1: \beta_j \neq 0$



10.167

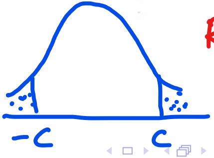
sig. level $= 0.05$

$df = n-k-1$

$= 526-1-1$

$= 524$

$c = 1.96$



Reject H_0.

Single Hypothesis - Single Parameter (cont.)

Using previous example

$H_0 : \beta_1 = 0.6$

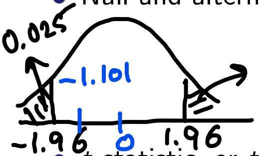
$H_1 : \beta_1 \neq 0.6$

$$t = \frac{\hat{\beta}_1 - 0.6}{se(\hat{\beta}_1)}$$

$= -1.101$

Testing Other Hypotheses about β_j

- Null and alternative (two-tailed test)



$$H_0 : \beta_j = a_j$$
$$H_1 : \beta_j \neq a_j$$

Sig. level = 0.05

$n-k-1 = 524$

- t statistic, or t ratio

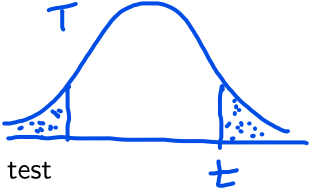
$$t = \frac{\hat{\beta}_j - a_j}{se(\hat{\beta}_j)}$$

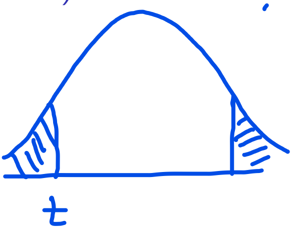
$c = 1.96$

Rej Ho if $|t|$ is suff. large
else fail to rej. Ho.

Fail to rej. Ho.

Single Hypothesis - Single Parameter (cont.)





p-value

- Two-tailed test
- t : t statistic, or t ratio
- T : t distributed random variable with $df = n - k - 1$

$$p\text{-value} = P(|T| > |t|)$$

- If $t > 0$, p-value $= 2P(T > t)$
- If $t < 0$, p-value $= 2P(T < t)$

Single Hypothesis - Single Parameter (cont.)

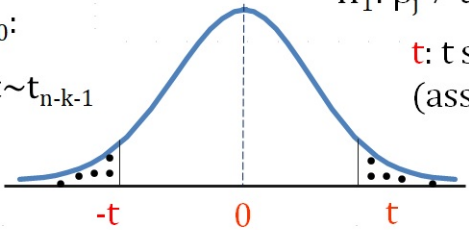
H_0: $\beta_j = a_j$

H_1: $\beta_j \neq a_j$

Under H_0:

$$\frac{\hat{\beta}_j - a_j}{se(\hat{\beta}_j)} = t \sim t_{n-k-1}$$

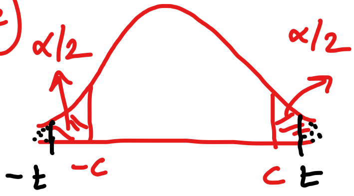
t: t statistic, or t ratio (assumed > 0)



Sum of tail areas = p-value = 2P(T > t) where T denotes a t distributed random variable with df = n-k-1

Single Hypothesis - Single Parameter (cont.)

α : $P(\text{rej. } H_0 \mid H_0 \text{ true})$



Reject H_0 if p-value is sufficiently small else fail to reject H_0

- Rejection rule
$$p\text{-value} < \alpha$$
- *Economic* or *practical* significance: related to $\hat{\beta}_j$
- *Statistical* significance: related to $t_{\hat{\beta}_j}$

Single Hypothesis - Single Parameter (cont.)

$wage = \beta_0 + \beta_1 educ + u$

$se(\hat{\beta}_0) = 0.685$

$\hat{\beta}_0 = -0.905$

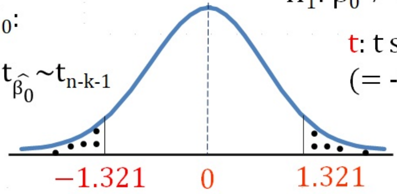
$H_0: \beta_0 = 0$

$H_0: \beta_0 = 0$

$H_1: \beta_0 \neq 0$

Under H_0:

$$\frac{\hat{\beta}_0}{se(\hat{\beta}_0)} \equiv t_{\hat{\beta}_0} \sim t_{n-k-1}$$



t: t statistic, or t ratio (= -1.321)

$t_{\hat{\beta}_0} = -1.321$

Sum of tail areas = p-value = $2P(T < -1.321) = 0.1868$

$n-k-1 = 524$

`https://www.npr.org/sections/13.7/2014/06/02/318212713/science-trust-and-psychology-in-crisis`

$\alpha = 0.05$

Fail to reject H_0.

p-value $= 2 \times 0.0934$
$= 0.1868$

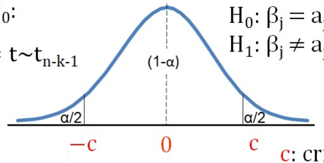
Single Hypothesis - Single Parameter (cont.)

Confidence interval

- Two-tailed test

Under H_0:

$$\frac{\hat{\beta}_j - a_j}{se(\hat{\beta}_j)} = t \sim t_{n-k-1}$$



H_0: $\beta_j = a_j$
H_1: $\beta_j \neq a_j$

c: critical value for t

Middle area = confidence level = P(fail to rej. $H_0 | H_0$ true)

Sum of tail areas = significance level = P(rej. $H_0 | H_0$ true)

$$P\left(\frac{\hat{\beta}_j - aj}{se(\hat{\beta}_j)}\right) \text{lies between } \pm c =$$

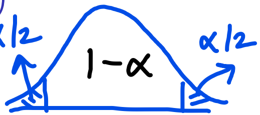
$$P(\hat{\beta}_j - a_j) \text{ lies between } \pm c \cdot se(\hat{\beta}_j) =$$

$$P(a_j) \text{ lies between } \hat{\beta}_j \pm c \cdot se(\hat{\beta}_j) \Rightarrow$$

P(fail to rej. $H_0 | H_0$ true).

Single Hypothesis - Single Parameter (cont.)

α : $P(\text{rej. } H_0 \mid H_0 \text{ true})$



- Significance level
- Confidence level : $1-\alpha = P(\text{not rej. } H_0 \mid H_0 \text{ true})$

At $\alpha = 0.05$, $1-\alpha = 0.95$ fail to reject H_0 for any hypothesized value in this interval.

- Null and alternative (two-tailed test)

$\log(wage) = \beta_0 + \beta_1 educ + \beta_2 exper + \beta_3 tenure + u$

$$H_0 : \beta_j = a_j$$
$$H_1 : \beta_j \neq a_j$$

- Reject H_0 if a_j lies beyond $\hat{\beta}_j \pm c \cdot se(\hat{\beta}_j)$
- Confidence interval for β_j

CI: $0.092 \pm 1.96 \times 0.007$ $\hat{\beta}_j \pm c \cdot se(\hat{\beta}_j)$ $[0.078, 0.106]$

$\hat{\beta}_1 = 0.092$

$se(\hat{\beta}_1) = 0.007$

95% CI for β_1 : $1-\alpha = 0.95$, $\alpha = 0.05$

c for $t_{526-3-1}$ or t_{522} $= 1.96$

Single Hypothesis - Multiple Parameters

$\log(wage) = \beta_0 + \beta_1\, educ + \beta_2\, exper + \beta_3\, tenure + u$

$\hat{\beta}_1 = 0.092, \quad se(\hat{\beta}_1) = 0.007 \qquad \hat{\beta}_2 = 0.004, \quad se(\hat{\beta}_2) = 0.002$

- Null and alternative (two-tailed test)

$H_0 : \beta_1 = \beta_2$

$H_1 : \beta_1 \neq \beta_2$

$$H_0 : \beta_j = \beta_l$$
$$H_1 : \beta_j \neq \beta_l$$

- Accordingly

Stata: lincom educ − exper

$$H_0 : \beta_j - \beta_l = 0$$
$$H_1 : \beta_j - \beta_l \neq 0$$

- t statistic, or t ratio

$$t = \frac{\hat{\beta}_j - \hat{\beta}_l}{se(\hat{\beta}_j - \hat{\beta}_l)}$$

$t_{522} = 12.59$

$\alpha = 0.05 \Rightarrow$ Rej. H_0.

Multiple Hypotheses

- Model

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_k x_k + u$$

- Null and alternative (two-tailed test)

H_0 : $\beta_1 = 0$, $\beta_2 = 0$

H_1 : at least one of these $\neq 0$

at least β_1 or $\beta_2 \neq 0$

Multiple Hypotheses (cont.)

- Unrestricted model

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \dots + \beta_k x_k + u$$

- Restricted model

$$y = \beta_0 + \beta_3 x_3 + \dots + \beta_k x_k + u$$

Multiple Hypotheses (cont.)

- Check for change in fit due to restrictions
- SSR_{ur}: SSR from unrestricted model
- SSR_r: " " restricted "

$$SSR_r \geq SSR_{ur}$$

- F statistic

$$F = \frac{\left[(SSR_r - SSR_{ur})/q\right]}{\left[SSR_{ur}/(n-k-1)\right]}$$

- Under H_0, $F \sim F_{q,n-k-1}$
  - q : numerator df (no. of restrictions or β coeffs. tested)
  - $n-k-1$:

denominator df

Multiple Hypotheses (cont.)

- Reject H_0 if F is sufficiently large else fail to rej. H_0.
- Rejection rule

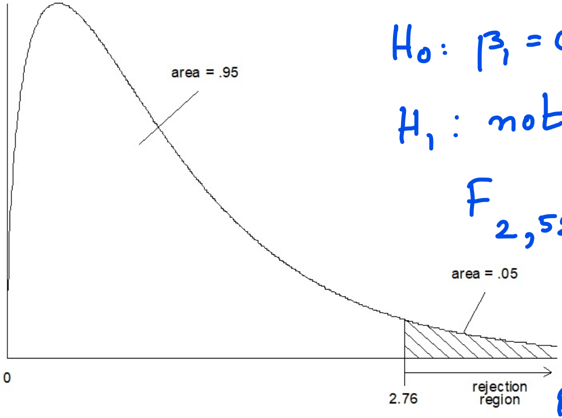
$$F > c$$

- Critical values - Tables G.3a, G.3b, and G.3c
- t^2_{n-k-1} is identical to $F_{1,n-k-1}$

Multiple Hypotheses (cont.)

$\log(wage) = \beta_0 + \beta_1 educ + \beta_2 exper + \beta_3 tenure + u$

- Example
  - $q = 3$ and $n - k - 1 = 60$
  - Critical value for $\alpha = 0.05$ is 2.76



$H_0: \beta_1 = 0, \beta_2 = 0$

$H_1:$ not H_0

$F_{2,522} = 80.148$

$\alpha = 0.05$

crit. value $= 3$

Reject H_0.

Multiple Hypotheses (cont.)

R-squared form of the F statistic

- Since

$$R^2 = 1 - \frac{SSR}{SST}$$

- We have

$$R_r^2 = 1 - \frac{SSR_r}{SST} \text{ and } R_{ur}^2 = 1 - \frac{SSR_{ur}}{SST}$$

- Accordingly

$$SSR_r = (1 - R_r^2)\,SST$$
$$SSR_{ur} = (1 - R_{ur}^2)\,SST$$

Multiple Hypotheses (cont.)

$\log(wage) = \beta_0 + \beta_1 educ + \beta_2 exper + \beta_3 tenure + u$

$H_0: \beta_1 = 0 \;\&\; \beta_2 = 0$

$H_1:$ not H_0

- F statistic

$$F = \frac{[(SSR_r - SSR_{ur})/q]}{[SSR_{ur}/(n-k-1)]}$$

- Alternatively

$$F = \frac{[(R^2_{ur} - R^2_r)/q]}{[(1 - R^2_{ur})/(n-k-1)]}$$

$R^2_{ur} = 0.316$

$R^2_r = 0.106$

$q = 2,\; n-k-1 = 522$

$F = 80.148$

Multiple Hypotheses (cont.)

$\log(wage) = \beta_0 + \beta_1 educ + \beta_2 exper + \beta_3 tenure + u$

The F statistic for overall significance of a regression

- Model

$H_0: \beta_1 = 0, \beta_2 = 0,$ $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_k x_k + 0$

- Null and alternative (two-tailed test)

$\beta_3 = 0$

H_1 : not H_0

$$H_0: \beta_1 = 0, \beta_2 = 0, \ldots, \beta_k = 0$$
$$H_1: \text{at least one of } \beta_1, \beta_2, \ldots, \beta_k \neq 0$$

- Here, $R^2_{ur} = R^2$ and $R^2_r = 0$
- F statistic

$q = 3$

$n-k-1 = 522$

$$F = \frac{R^2/k}{(1-R^2)/(n-k-1)}$$

$R^2_{ur} = 0.316$

$R^2_r = 0$

$F = 80.39$